# Physics 606 Final Exam

**Please be well-organized, and show all significant steps clearly in all problems.**

**You are graded on your work.**

**An answer, even if correct, will receive zero credit unless it is obtained via the work shown.**

Do all your work on the blank sheets provided, writing your name clearly, and turn them in stapled together. You may keep these questions.

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$$\psi_{\vec{k}}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} + \frac{e^{ikr}}{r} f(\vec{k}',\vec{k}) \quad , \quad f(\vec{k}',\vec{k}) = -\frac{m}{2\pi\hbar^2}\int d^3r'\, e^{-i\vec{k}'\cdot\vec{r}'} V(\vec{r}')\psi_{\vec{k}}(\vec{r}') \quad , \quad \frac{d\sigma}{d\Omega} = \left|f(\vec{k}',\vec{k})\right|^2$$

$$f(\theta) = -\frac{2m}{\hbar^2}\frac{1}{q}\int_0^\infty dr\, r\sin qr\, V(r) \quad , \quad \vec{q} = \vec{k}' - \vec{k} \quad , \quad q = 2k\sin(\theta/2)$$

for ground state of hydrogen, $\psi_0(\vec{r}) = \frac{1}{\sqrt{4\pi}}\frac{2}{a_0^{3/2}}e^{-r/a_0}$ *and* $E_0 = -k\frac{e^2}{2a_0} \approx -13.6\text{ eV}$ , $a_0 = \frac{\hbar}{mke^2}$

$$\Gamma(z) = \int_0^\infty du\, e^{-u} u^{z-1} \quad \text{or} \quad \int_0^\infty du\, e^{-u} u^n = \Gamma(n+1) = n!$$

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1. (a) (10) For an electron in its ground state in hydrogen, calculate the value of $r = |\vec{r}|$ for which the probability is highest. How does this relate to the orbit of an electron in the Bohr model?

(b) (10) Again for the ground state, calculate the expectation value of $r^2$, $\langle r^2 \rangle$.

From this infer the value of $\langle x^2 \rangle$ for a single rectangular coordinate.

(c) (10) Calculate the expectation value of $p_r^{\,2}$, $\langle p_r^{\,2} \rangle$, where $p_r = -i\hbar\left(\dfrac{\partial}{\partial r} + \dfrac{1}{r}\right)$, in terms of constants.

Then show that the uncertainties for a particular rectangular coordinate, $\Delta x$ and $\Delta p_x$, satisfy the Heisenberg uncertainty relation.

2. (20) The proton in a hydrogen atom is not really a point charge, and its diameter is roughly $1\,\mathrm{fermi} = 10^{-15}\,\mathrm{m} \approx 10^{-5} \times 2\,\mathrm{a}_0$. (The actual size and composition of the proton is a current area of research.) To crudely estimate what might be the effect of this finite size, let us model the proton by a hollow shell of charge of radius

$$R = 10^{-5}\ a_0$$

where $a_0$ is the Bohr radius. Then the potential energy of an electron is

$$V(r) = -k\frac{e^2}{R} \quad \text{for } r < R$$

$$V(r) = -k\frac{e^2}{r} \quad \text{for } r > R$$

where $k$ is the Coulomb's law constant.

For the $1s$ ground state of hydrogen, calculate the first-order change $\Delta E_{1s}$ in the energy due to the finite size of the proton in this model.

You may make the approximation $R \ll a_0$ where appropriate.

Give your answer in terms of $a_0$, $e$, and $k$.

Also give a roughly approximate numerical answer, in eV.

3. (a) (10) In the first Born approximation, calculate the scattering amplitude $f(\theta)$ and the differential cross section $\frac{d\sigma}{d\Omega}$ for the potential

$$V(r) = a\frac{e^{-Kr}}{r}$$

where $a$ and $K$ are constants. Express your result first in terms of $q^2 + K^2$, and then substitute for the magnitude $q$ of the scattering wavevector to write it in a form related to the expression in part (b) below.

(b) (5) Relate your result to the classical Rutherford scattering cross-section

$$\frac{d\sigma}{d\Omega} = \left(\frac{ZZ'ke^2}{4E_{\bar{k}}\sin^2(\theta/2)}\right)^2 \quad \text{for the Coulomb potential} \quad V(r) = k\frac{(Ze)(Z'e)}{r} \quad .$$

(c) (5) Show that the $d\sigma / d\Omega$ **of part (a)** is a constant (independent of both $k$ and $\theta$) in the limit of low momentum transfer $q$.

4. Consider a hydrogen atom which is in its ground state for $t < 0$. For $t > 0$ it is subjected to a spatially uniform electric field

$$\mathcal{E}_0 e^{-t/\tau}$$

which is directed along the $z$ axis. Using first-order time-dependent perturbation theory, let us calculate the probability of finding the atom in the excited state with $n = 2,\ \ell = 1,\ m = 0$ after a time $t$ has elapsed.

First, however, let us obtain the expression which gives the probability. The eigenstates of the unperturbed Hamiltonian are $|n\rangle$: $H_0|n\rangle = \varepsilon_n|n\rangle$. The full Hamiltonian in the original Schrödinger picture is $H = H_0 + V_t$, but recall that the time dependence of the state $|\psi(t)\rangle$ in the interaction picture arises only from

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = V(t)|\psi(t)\rangle$$
$$V(t) \equiv e^{iH_0 t/\hbar}\, V_t\, e^{-iH_0 t/\hbar} \quad .$$

(a) (1) Show that the solution of this equation to first order is

$$|\psi(t)\rangle = |\psi(t_0)\rangle + \frac{1}{i\hbar}\int_{t_0}^{t} dt'\, V(t')|\psi(t_0)\rangle .$$

(b) (3) If the system is in an initial state $|i\rangle$ at time $t_0 = 0$, show that the probability that it is in a different state $|n\rangle$ at time $t$ is

$$P_{i\to n}(t) = \left|\frac{1}{i\hbar}\int_0^t dt'\, e^{i(\varepsilon_n-\varepsilon_i)t'/\hbar}\,\langle n|V_{t'}|i\rangle\right|^2 .$$

(c) (1) Write down the perturbing Hamiltonian $V_t$ for the electron in a hydrogen atom subjected to an electric field with $\mathcal{E}_z = \mathcal{E}_0 e^{-t/\tau}$ and $\mathcal{E}_x = \mathcal{E}_y = 0$. Recall that $\text{force} = -\dfrac{dV_t}{dz}$, and that $z = r\cos\theta$.

(d) (10) Calculate the matrix element $\langle n|V_t|i\rangle$ for $|i\rangle = |100\rangle$ and $|n\rangle = |210\rangle$, with the states labeled as usual by the quantum numbers $|n\,\ell\,m\rangle$. The wavefunctions are given by

$$R_{10} = \frac{2}{a_0^{3/2}}e^{-r/a_0} \quad , \quad Y_{oo} = \frac{1}{\sqrt{4\pi}} \quad , \quad R_{21} = \frac{1}{(2a_0)^{3/2}}\frac{r}{a_0\sqrt{3}}e^{-r/2a_0} \quad , \quad Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta .$$

(e) (10) Calculate the probability $P_{100\to 210}(t)$ that an electron initially in the ground state $|100\rangle$ will be found in the excited state $|210\rangle$ after a time $t$ has elapsed. Give your answer in terms of $t$, $\tau$, $\mathcal{E}_0$, $e$, $\hbar$, and

$$\omega \equiv \frac{E_2 - E_1}{\hbar} .$$

(f) (5) Calculate the probability after a long time has elapsed, $P_{100\to 210}(t\to\infty)$.

5. (5 extra credit) We considered the decay of a state as the system undergoes transitions to other states.

Give an example in each of the following areas, with specifics.

(i) condensed matter physics or electrical engineering

(ii) atomic physics or quantum optics

(iii) nuclear physics

(iv) high energy physics

(v) astrophysics